ED-459

M.A./M.Sc. 2nd Semester
Examination, May-June 2021

MATHEMATICS

Paper - I

Advanced Abstract Algebra-II

Time : Three Hours] [*Maximum Marks* : 80

Note : Answer any **two** parts from each question. All questions carry equal marks.

Unit-I

1. (*a*) Show that Rc is large for $c \neq 0$, $c \in R$ and R is a noetherian integral domain.

(*b*) Show that for any noetherian ring R each ideal contains a finite product of prime ideals.

(*c*) Can we prove every submodule of a noetherian module is finitely generated ? How ?

Unit-II

2. (*a*) Prove that regular elements in $A(V)$ form a group.

(*b*) In V define T by

$$\left(\sum_{n=0}^{3} \alpha_n x^n\right) T = \sum_{n=0}^{3} \alpha_n (x+1)^n .$$

Compute the matrix of T in the basis $(1, 1+x, 1+x^2, 1+x^3)$.

(*c*) If V is an n-dimensional vector space over F, then for given $T \in A(V)$ there exists a non-trivial polynomial $g(x) \in F[x]$ of degree at most n^2, such that $g(T) = 0$. Prove it.

Unit-III

3. (*a*) Show that the elements S and T in $A_F(V)$ are similar in $A_F(V)$ if and only if they have the same elementary divisors.

(*b*) Find all possible rational canonical forms and elementary divisors for the 6×6 matrix in F_6 having $(x-1)(x^2+1)^2$ as minimal polynomial.

(*c*) Define nilpotent transformation and show that ST-TS is nilpotent iff $S, T \in A_F(V)$, ST-TS commutes with S and F is of characteristics zero.

Unit-IV

4. (*a*) Obtain the Smith normal form and rank for

$$\begin{bmatrix} -(x+3) & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -(x+2) \end{bmatrix}.$$

(*b*) Show that if V is a finite dimensional vector space over F, then V is a finitely generated $F[x]$ module.

(*c*) Let $T \in \mathrm{Hom}_F(V, V)$. Then show that there exists a basis of V with respect to which the matrix of T is $A = \mathrm{diag}(B_1, B_2, ...Br)$ where B_i is the companion matrix of a certain unique polynomial $f_i(x)$, $i = 1, 2, ... r$ such that $f_1(x) \mid f_2(x) \mid ..... \mid f_r(x)$.

Unit-V

5. (*a*) Find the rational canonical form of a matrix whose invariant factors are $x+2$, $x^2 - x - 6$, $x^3 - 2x^2 - 5x + 6$.

(*b*) Find Jordan canonical form of a matrix with characteristics polynomial $p(x) = (x-1)^2 (x+1)$.

(*c*) Find invariant factors, elementary divisors and Jordan canonical form of the matrix

$$\begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}.$$

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