FD-762

M.A./M.Sc. 4th Semester
Examination, May-June 2022

MATHEMATICS

Paper - I

Functional Analysis-II

Time : Three Hours] [*Maximum Marks* : 80

Note : Answer any **two** parts from each question. All questions carry equal marks.

Unit-I

1. (*a*) If B and B' are Banach spaces and if T is a continuous linear transformation of B onto B', then prove that the image of each open space centred on the origin in B contains an open sphere centred on the origin in B'.

(*b*) State and prove uniform boundedness theorem.

(*c*) Show that a closed linear map T mapping a Banach space X into a Banach space Y is continuous.

Unit-II

2. (*a*) State and prove Hahn-Banach theorem for real linear space.

(*b*) Let X and Y are Banach spaces and $T \in B(X, Y)$. Then prove that $R(T)$ is closed if and only if $R(T^*)$ is closed.

(*c*) Let X be a normed spaces. Then show that X is reflexive if and only if every bounded sequence in X has a weak convergent subsequence.

Unit-III

3. (*a*) If X is an inner product space, then show that $\sqrt{\langle x, x \rangle}$ has the properties of a norm.

(*b*) Prove that every orthogonal set in a Hilbert space is contained in some complete orthogonal set. Further every non-zero Hilbert space contains a complete orthogonal set.

(*c*) If M and N are closed linear subspace of a Hilbert space H such that $M \perp N$, then prove that linear subspace $M + N$ is closed.

Unit-IV

4. (*a*) Let y be a fixed vector in a Hilbert space H and let f_y be a scalar value function on H defined by

$$f_y(x) = \langle x, y \rangle \, \forall x \in H$$

then show that f_y is a functional in H^* i.e. f_y is a continuous linear functional on H and $\| y \| = \| f_y \|$.

(*b*) State and prove Projection theorem.

(*c*) Show that the adjoint operation is one-to-one onto as a mapping of $B(H)$ into itself and $\| T^*T \| = \| T \|^2$.

Unit-V

5. (*a*) Let T be a normal operator on a Hilbert space H and p be a polynomial with complex coefficients, then show that the operator $p(T)$ is normal.

(*b*) If $P_1, P_2, ..., P_n$ are the projections on closed linear subspaces $M_1, M_2, ..., M_n$ of a Hilbert space H, then show that $P = P_1 + P_2 + ... + P_n$ is a projection if and only if P_is are paire wise orthogonal (in the sense that $P_i P_j = 0$ whenever $i \neq j$). Also then P is the projection on $M = M_1 + M_2 + M_3 + ..., M_n$.

(*c*) Show that if T is a positive operator on a Hilbert space H, then $I + T$ is non-singular.
