Roll No. ....

DD-459

M. A./M. Sc. (Second Semester) EXAMINATION, May/June, 2020

MATHEMATICS

Paper First

(Advanced Abstract Algebra—II)

Time : Three Hours

Maximum Marks : 80

Note : Attempt any *two* parts from each question. All questions carry equal marks.

Unit—I

1. (a) Let M be a (finitely generated unital) free R-module with a basis $\{e_1, e_2, ... e_n\}$, then show that $M \simeq R^n$.

(b) Prove that R-module M is noetherian if and only if every submodule of M is finitely generated.

(c) State and prove Wedderburn-Artin theorem.

Unit—II

2. (a) For $T \in \text{Hom}(U, V)$, $\alpha \in F$, then show that $\alpha T \in \text{Hom}(U, V)$.

(b) Let A be an algebra with unit element, over F. Then show that A is isomorphic to a subalgebra of A(V) for some vector space V over F.

(c) Let U and V be vector spaces over a field F, then show that $Hom_F(U, V) \simeq F^{m \times n}$ as vector space over F.

Unit—III

3. (a) Show that if $W \subset V$ be invariant under $T \in A(V)$, then T induces a linear transformation T on the quotient space V/W, defined by $(V + W)\bar{T} = VT + W$. If T satisfies the polynomial $q(x) \in F[x]$, then $\bar{T}$ also satisfy $q(x)$. If $P_1(x)$ is the minimal polynomial for $\bar{T}$ over F and if $P(x)$ is that for T then $p_1(x) \mid p(x)$.

(b) Let the linear transformation $T \in AF(V)$ be nilpotent, then show that $\alpha_0 + \alpha_1 T + \cdots + \alpha_m T^m$, where $\alpha i \in F, 0 \le i \le m$ is invertible if $\alpha_0 \neq 0$.

(c) Show that two nilpotent linear transformations S, $T \in A(V)$ are similar if and only if they have the same invariants.

Unit—IV

4. (a) Obtain the Smith normal form and rank for the following matrix over a PID R :

$$\begin{bmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ 2 & -4 & -1 \end{bmatrix}, R = Z$$

(b) Obtain the Smith normal form and rank for the following matrix over PID R :

$$\begin{bmatrix} -x-3 & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -x-2 \end{bmatrix}, \text{ where } R = Q[x]$$

(c) Find the abelian group generated by (x_1, x_2, x_3) subject to $5x_1 + 9x_2 + 5x_3 = 0$, $2x_1 + 4x_2 + 2x_3 = 0$, $x_1 + x_2 - 3x_3 = 0$.

Unit—V

5. (a) Reduce the following matrix A to a rational canonical form :

$$\begin{bmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

(b) Let V be a finite dimensional vector space over a field F, and let $T \in \text{Hom}\, F(V, V)$. Suppose $f(x) = g(x)h(x)$ is a factorization of $f(x)$ in $f(x)$ such that gcd $(g(x), h(x)) = 1$, then show that $f(T) = \hat{0}$ if and only if $V = \ker g(T) \textcircled{f} \ker h(J)$.

(c) Find invariant factors, elementary divisors and the Jordan canonical form of the following matix :

$$\begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}$$